Correction of solution for linear and nonlinear boundary value problems

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We consider the boundary value problem in the general form

$$z'(x) = f(x, z(x)), \quad a \leq x \leq b, \tag{1}$$

$$g(z(a), z(b)) = 0, \tag{2}$$

and we suppose a solution of (1)-(2) to exist unique or isolated.
The shooting method is employed on the mesh $x_i = x_0 + ih, i = 0, 1, \ldots, n$, with $x_0 = a$, the constant steplength $h = (b-a)/n$ and a number of shooting subinterval $m = n/l$ for some integer l. Thus the shooting nodes are $x_0, x_l, \ldots, x_{(m-1)l}, x_n$ and an approximation z_{ji} to $z_j(x_i)$ $j = 1, 2, \ldots, m, i = (j-1)l, \ldots, jl$, is obtained with the global discretization error (GDE) given by $e_j(x_i) = z_j(x_i) - z_{ji}$. A general theorem has been proved giving the following formula for the GDE

$$e_i = r_i - \Gamma_i r_N, \tag{3}$$

where $e_i = (e_1(x_{i_1})^T, \ldots, e_m(x_{i_m})^T)^T$, while r_i is the block vector of order m containing the discretization errors of the solutions of the initial value problems (IVPs), associated with the shooting procedure, computed in the internal points and r_N similarly to r_i but computed in the last points of shooting subintervals. Γ_i is a block matrix of order m whose form is depending on whether the problem is linear or nonlinear.
In integrating the IVPs in the linear case the boundary value method has been used because it is well suited in order to compute the r_i and r_N vectors when an estimation of the local truncation errors of the linear multistep formulas employed is given.
The procedure previously decribed can be extended to the regular Sturm-Liouville problem

$$z'(x) = H(x, \lambda) z(x), \quad a \leq x \leq b, \tag{4}$$

$$Az(a) + Bz(b) = 0, \tag{5}$$

to approximate an eigenfunction $z(x)$ and the associated eigenvalue λ. In (4)-(5) the matrices H, A, B are of order 2, $z(x)$ is a 2-vector, while, for the constant matrices A and B, the condition $rank(A, B) = 2$ is required.
The formula (3) is particularly easy to be used to correct the approximation of the eigenfunction. Consequently the scalar equation we have to solve to approximate the eigenvalue λ is near to the 'exact' form. Thus the rootfinder procedure can be efficiently applied. Furthermore an extrapolation technique, not expensive by respect to the known ones, is described.
Some numerical experiments dealing with classical problems confirm the theoretical results.

References

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