Hyperbolic Systems of Conservation Laws with Stiff Diffusive Relaxation

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Hyperbolic systems of conservation laws with relaxation terms occour in many physical situations: kinetic theory of gases, gas flows with relaxation, multiphase and phase transition, turbulence, water waves, viscoelasticity and reactive flows. The study of the zero relaxation limit for such systems has cought much interest, both from a theorical and numerical point of view [1, 4, 6, 10]. The small relaxation limits of such systems are governed by reduced systems of parabolic or hyperbolic type. In general the understanding of basic wave pattern is difficult to achieve and standard high resolution methods fail to describe the right asymptotic behaviour unless the small relaxation rate is numerically resolved. In this work we are mainly concerned with numerical methods for hyperbolic systems with relaxation in the diffusive scaling. A simple prototype of hyperbolic system with diffusive relaxation is given by a generalization of the Carleman's model [5] of the Boltzmann equation. The evolution equations in macroscopic form are nonlinear diffusion equations including the classical porous media equation (see also [2, 3] for more complex cases). Here we consider the more general case in which the limit equation is a nonlinear diffusion problem. In particular, we would like to investigate, from a numerical point of view, the possibility of obtaining the macroscopic behavior described by the equilibrium system by solving the original relaxated hyperbolic system with coarse grids and avoiding spurios numerical solutions. A second order scheme is developed in order to study the numerical passage between different regime. This scheme combines an explicit high-order Godunov method with implicit central difference for the diffusive source term in the stiff regime for system, that is to say when the relaxation parameter becomes small. A Runge-Kutta method is used to achieve second order accuracy in time [2, 7, 8, 9]. We also present a relaxation approximation to general systems of nonlinear reaction-diffusion equations which permits us to develop a class of stable numerical schemes for the original systems. In particular we show that for many systems of reaction-diffusion equations the relaxing systems have a clear kinetic interpretation as generalized Carleman's models with source terms. Specific relaxation approximation are pointed out for more general reaction-diffusion systems also in multidimensional case. Finally, numerical results are presented for a generalized Fisher's equation including test for travelling waves solution.

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