Conditioning and Stability Problems

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In [1] a general method for approximating polynomial solutions of second-order linear homogeneous differential equations with polynomial coefficients is applied to the case of the families of differential equations defining the Generalized Bessel Polynomials (GBPs), and an algorithm is derived for simultaneously finding their zeros.

Then a comparison with several alternative methods and respective algorithms is carried out. Known sharp results about asymptotic properties and the location of the zeros of GBPs are used both to define the algorithm presented in [1] and to judge of the accuracy of the outputs yielded by all the compared algorithms. The comparison shows that the computational problem of approximating the zeros of the GBPs is not an easy matter at all and that the only algorithm able to give an accurate solution seems to be the one presented in the paper.

Questions of condition and stability related to basic Numerical Analysis problems like the computation of the eigenvalues of Jacobi matrices or the solution of algebraic equations are faced and it is shown that the method and the algorithm proposed in this article can be viewed as a successful attempt to overcome difficulties arising in the solution of those general problems by exploiting the special features of the case under consideration.

Conjectures suggested by tests carried out using the algorithm and concerning the location of the zeros in the case of complex values of the parameters which the GBPs depend on are also stated.

In [2] we deal with the sequence $\xi = \{\xi_n\}_{n=0,1,\dots}$ defined by the three-term recurrence

$$n = 4\,\xi_n(\xi_{n-1} + \xi_n + \xi_{n+1}), \quad n = 1, 2, \dots$$

and by the initial conditions

$$\xi_0 = 0, \quad \xi_1 = \Gamma(3/4)/\Gamma(1/4).$$

Our interest in this matter can be motivated as follows.

First of all we point out that the above recurrence and its solution ξ are closely related to the most investigated Freud system of polynomials. The ξ_n's play a central role in many questions. For instance, the greatest zero of the polynomial p_n of the system can be expressed in terms of ξ_n, so that any information about the asymptotic behavior of the solution ξ can be used to get estimates of the greatest zeros of the p_n's. Another interesting application is shown in [3]. Here the ξ_n's are used to study electrostatic interacting particle models associated with Freud weights, showing that it is possible to express the equilibrium energy of the model entirely in terms of the ξ_n's.

A second reason of interest in the subject is that the solution ξ is the unique nonnegative solution of the recurrence. This adds interest to the matter as it implies that the computational problem and the algorithm one naturally derives from the three-term recurrence will necessarily be, respectively, ill-conditioned and unstable.

Owing to its importance, the sequence ξ has been studied by many authors. Properties of ξ have been shown.

In this paper we show further properties of ξ. First we establish bounds for the departure of ξ from the sequence which it asymptotically converges to. Then we prove that ξ is an increasing sequence.

References

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