Stable Numerical Methods for Nonlinear Neutral Delay Differential Equations in Hale's form

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We consider nonlinear Neutral Delay Differential Equations (NDDEs) in Hale's form

$$\frac{d}{dt}(y(t) - G(t, y(t-\tau(t)))) = F(t, y(t), y(t-\tau(t))), \qquad t \geq t_0, \tag{1}$$

with initial condition

$$y(t) = g(t), \qquad t \leq t_0, \tag{2}$$

where F, G, g and $\tau(t) \leq t$ are given functions. Different reasons stimulated our interest for NDDEs in Hale's form. The equations (1) have been extensively studied in recent years since many phenomena in applied science can be modelled by them (see for instance the book of Hale and Verduyn [1]). Moreover, since one can take advantage of their special form to derive easier results on solvability and stability of the analitic solutions, we wonder whether we could benefit by the special form of (1) also in designing numerical methods and in studying their stability properties. Suppose that the equations (1) can be rewritten in the general form

$$y'(t) = F_1(t, y(t), y(t-\tau(t)), y'(t-\tau(t))), \qquad t \geq t_0, \tag{3}$$

where

$$F_1(t, y, v, z) = F(t, y, v) + \frac{\partial}{\partial t}G(t, v) + \frac{\partial}{\partial v}G(t, v)z(1-\tau'(t)).$$

with the same initial condition (2) and that the solution $y(t)$ and the functions F and G are smooth. Then the right hand side of (1) has one degree more of continuity of the right hand side of (3). Since the solutions of NDDEs are not smooth in general, a slight increase of smoothness of the functions is important for the accuracy of the numerical method. For instance, we could use a second order method with linear interpolation for the retarded part without taking care of discontinuity points, mantaining the order two. Other remarks on the advantage of using (1) instead of (3) can be found in the recent paper of Liu [2]. Our aim is to focus the attention on stability of numerical schemes. As usual, before deriving "good" numerical methods, we need to understand the stability properties of the analitic solution. Thus we study respectively the contractivity and the asymptotic stability properties of the solutions of (1). First, by a suitable reformulation of (1), we eliminate the dependence on the derivative of the solution on the past values and we obtain a differential-algebraic system. Then, by using the well-known theory of stability with respect a forcing term (see Zennaro [3]) to the ordinary differential part, we can derive sufficient conditions both for boundedness and asymptotic stability of solutions.

This theory reveals itself to be an interesting alternative approach to investigate stability properties. It allows us to treat nonlinear problems with a wide class of delay functions $\tau(t)$, for instance, we can also consider an oscillating deviating argument. Another advantage is that the results can be easily and directly reproduced for the numerical solutions. We introduce a numerical procedure based on continuous Runge-Kutta methods and we discuss the uniform order of approximation. Finally, we present numerical schemes preserving the stability properties of the underlying equation.

References

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