Solutions to the exercises in Lecture 8 by the Wuppertal team

Exercise 8.1

Prove the following equalities:

(i) $H_k'(x) = \sqrt{k}H_{k-1}(x) = xH_k(x) - \sqrt{k+1}H_{k+1}(x)$

(ii) $H_k''(x) - xH_k'(x) = -kH_k(x)$,

where $H_k = \frac{(-1)^k}{\sqrt{k!}} \mathrm{e}^{x^2/2} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathrm{e}^{-x^2/2}$ are the Hermite polynomials introduced in Definition 8.1.1.

Solution:

We will show the validity of the identities in (i) by directly computing derivatives. In order to do so, we note that, by using the Leibniz formula $(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$, we obtain:

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} \left(x \cdot \mathrm{e}^{-x^2/2} \right) = x \cdot \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathrm{e}^{-x^2/2} + k \cdot \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} \mathrm{e}^{-x^2/2}.$$

We start by showing that the second equality in (i) holds. By differentiating and by using the previously mentioned consequence of the Leibniz formula we see that

$$\begin{aligned} xH_k(x) - \sqrt{k+1}H_{k+1}(x) &= \frac{(-1)^k}{\sqrt{k!}} \mathrm{e}^{x^2/2} \cdot x \cdot \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathrm{e}^{-x^2/2} - \sqrt{k+1} \frac{(-1)^{k+1}}{\sqrt{(k+1)!}} \mathrm{e}^{x^2/2} \frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}} \mathrm{e}^{-x^2/2} \\ &= \frac{(-1)^k}{\sqrt{k!}} \mathrm{e}^{x^2/2} \left(x \cdot \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathrm{e}^{-x^2/2} - \frac{\mathrm{d}^k}{\mathrm{d}x^k} \left(x \cdot \mathrm{e}^{-x^2/2} \right) \right) \qquad (*) \\ &= \frac{(-1)^k}{\sqrt{k!}} \mathrm{e}^{x^2/2} \left(x \cdot \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathrm{e}^{-x^2/2} - \left(x \cdot \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathrm{e}^{-x^2/2} + k \cdot \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} \mathrm{e}^{-x^2/2} \right) \right) \\ &= \sqrt{k} \frac{(-1)^{k-1}}{\sqrt{(k-1)!}} \mathrm{e}^{x^2/2} \frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} \mathrm{e}^{-x^2/2} = \sqrt{k}H_{k-1}(x). \end{aligned}$$

To show the first equality in (i) we simply have to differentiate the Hermite polynomial H_k. We thus obtain

$$H_k'(x) = \frac{(-1)^k}{\sqrt{k!}} \left(x \cdot \mathrm{e}^{x^2/2} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathrm{e}^{-x^2/2} + \mathrm{e}^{x^2/2} \frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}} \mathrm{e}^{-x^2/2} \right).$$

Differentiating the second summand here, puts us in the situation $(*)$ above.

Next we turn to the proof of (ii). Using the already proven part (i) we conclude for $k \geq 2$ that

$$H_k''(x) = \sqrt{k}\sqrt{k-1}H_{k-2}(x) = \sqrt{k} \left(x \cdot H_{k-1}(x) - \sqrt{k}H_k(x) \right) \qquad (1)$$

as well as

$$xH_k'(x) = x\sqrt{k}H_{k-1}(x). \qquad (2)$$

By (1) and (2) equation (ii) becomes obvious. The identity (ii) for $k = 0, 1$ is trivial.

Exercise 8.2

After consultation with the organisers this exercis e has been discarded as the exercise was not well written.

Exercise 8.3

Verify that the family $\mathscr{F}_\gamma$ introduced in Definition 8.2.1 is a σ-algebra. Prove also that the measure γ, extended to $\mathscr{F}_\gamma$ by $\gamma(E) = \gamma(B_1) = \gamma(B_2)$ for E, B_1, B_2 as in Definition 8.2.1, is still a measure.

Solution:

First of all we prove, that the set $\mathscr{F}_\gamma$ introduced in Definition 8.2.1 is a σ-algebra. To do this we need to make sure, that $\mathscr{F}_\gamma$ has the following three properties:

1. $\emptyset \in \mathscr{F}_\gamma$.
2. For every set $A \in \mathscr{F}_\gamma$ the complement A^c is an element of the completion $\mathscr{F}_\gamma$.
3. For every sequence of sets $(A_n)_{n\in\mathbb{N}} \subset \mathscr{F}_\gamma$ the union $A := \bigcup\limits_{n\in\mathbb{N}} A_n$ is also an element of $\mathscr{F}_\gamma$.

Due to the fact that the σ-algebra $\mathscr{F}$ is a subset of $\mathscr{F}_\gamma$, the first property holds. We now continue with the proof of the second property. For a set $A \in \mathscr{F}_\gamma$ we know the existence of the sets $B_1, B_2 \in \mathscr{F}$, such that $B_1 \subset A \subset B_2$ and $\gamma(B_2 \setminus B_1) = 0$. As $B_1, B_2 \in \mathscr{F}$, the complements $B_1^c, B_2^c \in \mathscr{F}$. Furthermore, we obtain $B_2^c \subset A^c \subset B_1^c$ and $\gamma(B_1^c \setminus B_2^c) = 0$, because

$$\gamma(B_1^c \setminus B_2^c) = \gamma(B_1^c \setminus (X \setminus B_2)) = \gamma(B_1^c \cap B_2) = \gamma((X \setminus B_1) \cap B_2) = \gamma(B_2 \setminus B_1) = 0.$$

We now prove the third property. Let $(A_n)_{n\in\mathbb{N}} \subset \mathscr{F}_\gamma$ be a sequence of sets and A be the union of these sets $A := \bigcup\limits_{n\in\mathbb{N}} A_n$. We have to show that $A \in \mathscr{F}_\gamma$ holds. By assumption, we have that

$$\forall n \in \mathbb{N}\ \exists B_n, C_n \in \mathscr{F} \text{ such that } B_n \subset A_n \subset C_n \text{ and } \gamma(C_n \setminus B_n) = 0$$

and therefore

$$\bigcup_{n\in\mathbb{N}} B_n \subset \bigcup_{n\in\mathbb{N}} A_n \subset \bigcup_{n\in\mathbb{N}} C_n \text{ and } \gamma(\bigcup_{n\in\mathbb{N}} C_n \setminus \bigcup_{n\in\mathbb{N}} B_n) = 0$$

yields since

$$\bigcup_{n\in\mathbb{N}} C_n \setminus \bigcup_{n\in\mathbb{N}} B_n \bigcup_{n\in\mathbb{N}} (C_n \setminus B_n).$$

Thus, we get $A \in \mathscr{F}_\gamma$ and hence $\mathscr{F}_\gamma$ is a σ-algebra.
To prove that the extension of the measure γ on $\mathscr{F}_\gamma$ is a measure, we first will prove that the measure γ ist well-defined on $\mathscr{F}_\gamma$. Let therefore $E \in \mathscr{F}_\gamma$ be an arbitrary set. As E is an element of the completion, we have some sets $B_1, B_2 \in \mathscr{F}$ such that $B_1 \subset E \subset B_2$ and $\gamma(B_2 \setminus B_1) = 0$. Let us assume that there exist additionally two sets $C_1, C_2 \in \mathscr{F}$ such that $C_1 \subset E \subset C_2$ and $\gamma(C_2 \setminus C_1) = 0$. To prove that γ is well-defined, we have to check if the equations

$$\gamma(B_2) = \gamma(B_1) = \gamma(C_1) = \gamma(C_2) =: \gamma(E)$$

hold. Using the additivity of the measure, it is easy to see that $\gamma(B_1) = \gamma(B_2)$ and $\gamma(C_1) = \gamma(C_2)$. Indeed:

$$\gamma(C_2) = \gamma((C_2 \setminus C_1) \cup C_1) = \gamma(C_2 \setminus C_1) + \gamma(C_1) = \gamma(C_1) \tag{3}$$

It remains to show that $\gamma(B_1) = \gamma(C_1)$. We will give a proof for this assertion by contradiction. Therefore, we can assume without loss of generality that

$$\gamma(B_1) > \gamma(C_1). \tag{4}$$

From this equality, we conclude the following inequality:

$$\gamma(B_1) \overset{(4)}{>} \gamma(C_1) \overset{(3)}{=} \gamma(C_2) = \gamma((C_2 \setminus B_1) \cup B_1) = \gamma(C_2 \setminus B_1) + \gamma(B_1).$$

It follows then that $0 > \gamma(C_2 \setminus B_1)$, which is a contradiction to the non-negativity of the measure γ as the sets are all in $\mathscr{F}$.

In conclusion, we get $\gamma(B_2) = \gamma(B_1) = \gamma(C_1) = \gamma(C_2) =: \gamma(E)$. Therefore, γ is well-defined on $\mathscr{F}_\gamma$ and our next goal is to prove that the so extended γ is still a measure. (By the way it is trivial that the new γ is indeed an extension of the γ on $\mathscr{F}$.) This means we need to check whether the following properties

(i) γ is a non-negative function,

(ii) $\gamma(\emptyset) = 0$,

(iii) γ is σ-additivity,

hold.
With the equalities we have proved above it is obvious that these stated properties hold. Therefore, γ is indeed a measure on $\mathscr{F}_\gamma$. (It is also trivial to see that $\mathscr{F}_\gamma$ contains all (subsets of) γ-null sets. Whence comes the terminology "completion".)

Exercise 8.4

Prove that if A is a measurable set such that $A + rh_j = A$ up to γ-negligible sets with $r \in \mathbb{Q}$ and $\{h_j : j \in \mathbb{N}\}$ an orthonormal basis of H, then $\gamma(A) \in \{0, 1\}$.

Solution:

By Proposition 8.2.3 it is sufficient to show that $\gamma(A + h) = \gamma(A)$ holds for all $h \in H$. By the Cameron–Martin theorem (Theorem 3.1.5) for all $h \in H$ the measure $A \mapsto \gamma(A + h)$ is equivalent to γ with the density given by $\rho_h(x) = \exp(\hat{h}(x) - \frac{1}{2}|h|_H^2)$, where $\hat{h} = R_\gamma^{-1}h$ and $R_\gamma \colon X_\gamma^* \to H$ is an isometric isomorphism. As a consequence, by an application of Lebesgue's theorem, the map $h \mapsto \gamma(A + h)$ is continuous: If $h_n \to h$ in H, then

$$\gamma(A + h_n) = \int_X \exp(\hat{h_n}(x) - \frac{1}{2}|h_n|_H^2) d\gamma \to \gamma(A + h),$$

since the integrand converges pointwise, and an integrable majorant is yielded by Fernique's theorem (Theorem 2.3.1). For all $h \in H$ we have the representation $h = \sum_{j\in\mathbb{N}} \langle h, h_j \rangle h_j$. For each $j \in \mathbb{N}$ there is a sequence $(q_{j,k})_{k\in\mathbb{N}} \subset \mathbb{Q}$ such that $\langle h, h_j \rangle = \lim_{k\to\infty} q_{j,k}$. Hence we obtain for h the representation

$$h = \sum_{j\in\mathbb{N}} \lim_{k\to\infty} q_{j,k} h_j = \lim_{N\to\infty} \sum_{j=0}^{N} \Big(\lim_{k\to\infty} q_{j,k} h_j \Big) = \lim_{N\to\infty} \lim_{k\to\infty} \sum_{j=0}^{N} q_{j,k} h_j.$$

From the assumption it follows by induction on N that $A + \sum_{j=0}^{N} q_{j,k} h_j = A$ up to γ-negligible sets. Whence we obtain

$$\gamma(A + h) = \lim_{N\to\infty} \lim_{k\to\infty} \gamma\Big(A + \sum_{j=0}^{N} q_{j,k} h_j\Big) = \lim_{N\to\infty} \lim_{k\to\infty} \gamma(A) = \gamma(A).$$

Exercise 8.5

Prove that the functionals f defined in Example 8.3.2 enjoy the stated properties.

General comments

The goal of the exercise is to show that

$$f(x) = \sum_{n=1}^{\infty} c_n x_n, \quad x \in X, \tag{5}$$

defines a measurable linear functional on (X, γ) for the examples from Chapter 4, i.e.

(i) $(X,\gamma) = (\mathbb{R}^\infty, \bigotimes_{n\in\mathbb{N}} \gamma_1)$, $(c_n)_n \in \ell^2$ and with $x = (x_n)_{n\in\mathbb{N}}$ for $x \in \mathbb{R}^\infty$.

(ii) $(X,\gamma) = (X, \mathcal{N}(0,Q))$, where X is a Hilbert space and Q a self-adjoint, positive trace-class operator with eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$. Let $\{e_k : k \in \mathbb{N}\}$ be an orthonormal basis of eigenvectors of Q such that $Qe_k = \lambda_k e_k$ for all $k \in \mathbb{N}$. Here, x_n denotes $\langle x, e_n\rangle_X$ and the sequence $(c_n)_n$ is assumed to satisfy that $\sum_{n=1}^\infty c_n^2 \lambda_n < \infty$.

To show that f is a measurable linear functional, for both (i) and (ii), it remains to show that f is well-defined on a measurable subspace of full measure. Indeed, let us define

$$V = \{x \in X \,:\, \sum_{n=1}^\infty c_n x_n \text{ converges}\}.$$

It is obvious that V is a subspace and that $f|_V$ is well-defined and linear. For $n \in \mathbb{N}$ define

$$\mathcal{X}_n \colon X \to \mathbb{R}, x \mapsto x_n,$$

which is measurable for both of the above cases (i) and (ii). Hence, our goal is to show that

$$\mathcal{S}_N = \sum_{n=1}^N c_n \mathcal{X}_n \text{ converges pointwise } \gamma\text{-a.e. as } N \to \infty, \tag{6}$$

which would imply that

- V is measurable (since the measure is complete on $\mathcal{B}_\sigma$),
- $f = \lim_{N\to\infty} S_N$ is measurable,
- and that $\gamma(V) = 1$.

However, the space V is even Borel measurable, since

$$\begin{aligned} V &= \{x \in X : S_N(x) \text{ converges}\} \\ &= \{x \in X : \forall n \in \mathbb{N}\ \exists N_0 \in \mathbb{N}\ \forall (N, M \geq N_0) \colon |S_N(x) - S_M(x)| < \tfrac{1}{n}\} \\ &= \bigcap_{n\in\mathbb{N}} \bigcup_{N_0\in\mathbb{N}} \bigcap_{N\geq N_0} \bigcap_{M\geq N_0} \{x \in X : |S_N(x) - S_M(x)| < \tfrac{1}{n}\}, \end{aligned}$$

where the set $\{x \in X : |S_N(x) - S_M(x)| < \frac{1}{n}\}$ is measurable as the S_n's are measurable functions.

It remains to show (6). To do so, we observe that $\{\mathcal{Y}_k \colon k \in \mathbb{N}\}$, with $\mathcal{Y}_k = c_k \mathcal{X}_k$, is a sequence of independent random variables (this simply follows from the form of $\mathcal{X}_k$). Thus, by a results by P. Lévy, Theorem 1 below,

$$S_N \text{ converges pointwise } \gamma\text{-a.e.} \iff S_N \text{ converges in measure } [1].$$

Hence, it remains to show that S_N converges in measure, which clearly follows if we can show convergence in $L^2(X,\gamma)$.

[1] Note that for general sequences of measurable functions, only the implication "pointwise convergence a.e." $\implies$ "convergence in measure" holds.

Part (i)

Let $M, N \in \mathbb{N}$ and $M < N$ and $x \in \mathbb{R}^\infty$. Then, by a very similar computation as in Chapter 4 (p. 39), we deduce

$$
\begin{aligned}
\|S_N - S_M\|_{L^2(X,\gamma)} &= \int_{\mathbb{R}^\infty} \left| \sum_{k=M+1}^{N} c_k \mathcal{X}_k(x) \right|^2 \gamma(dx) \\
&= \int_{\mathbb{R}^\infty} \sum_{k=M+1}^{N} c_k^2 (\mathcal{X}_k(x))^2 \, \gamma(dx) + \int_{\mathbb{R}^\infty} \sum_{k \neq j} c_k c_j \mathcal{X}_k(x) \mathcal{X}_j(x) \, \gamma(dx) \\
&\overset{(a)}{=} \sum_{k=M+1}^{N} c_k^2 \int_{\mathbb{R}} y^2 \, \gamma_1(dy) + \sum_{k \neq j} c_k c_j \int_{\mathbb{R}} y \gamma_1(dy) \int_{\mathbb{R}} z \gamma_1(dz) \\
&\overset{(b)}{=} \sum_{k=M+1}^{N} c_k^2,
\end{aligned}
$$

where we used (a) that $\gamma = \bigotimes_{n\in\mathbb{N}} \gamma_1$ and the "push-forward" and (b) that $\gamma_1 = \mathcal{N}(0,1)$. Since $(c_n)_n \in \ell^2$, this shows that $(S_N)_N$ is Cauchy in $L^2(X,\gamma)$ and thus converges.
By the considerations above, this implies that S_N converges γ-a.e. and consequently that f defines a measurable linear functional.

If $(c_n)_n \in \mathbb{R}^\infty_c$, then $f : \mathbb{R}^\infty \to \mathbb{R}$ is continuous (moreover, $X^* = \mathbb{R}^\infty_c$), see Chapter 4). Thus, we assume that $(c_n)_n \in \ell^2 \setminus \mathbb{R}^\infty_c$. Define the sequence $(x^{(m)})_{m\in\mathbb{N}} \subset \mathbb{R}^\infty$ by

$$
x_n^{(m)} = \left\{ \begin{array}{cl} \frac{1}{c_n}\delta_{mn} & \text{if } c_n \neq 0 \\ 0 & \text{if } c_n = 0 \end{array} \right\} \qquad n, m \in \mathbb{N}.
$$

It is easy to see that $x^{(m)} \in \mathbb{R}^\infty_c$ and that

$$
f(x^{(m)}) = \left\{ \begin{array}{cl} 1 & \text{if } c_m \neq 0 \\ 0 & \text{if } c_m = 0 \end{array} \right\} \qquad m \in \mathbb{N}.
$$

for all m such that $c_m \neq 0$. Since $(c_n)_n \in \ell^2 \setminus \mathbb{R}^\infty_c$, we can find a subsequence $(m_k)_k$ such that $f(x^{(m_k)}) = 1$ for all $k \in \mathbb{N}$. However, since the topology on $\mathbb{R}^\infty$ equals the topology of pointwise convergence[2], we see that $x^{(m)}$ converges to $0 = (0, 0, ..)$ in $\mathbb{R}^\infty$. Therefore, f is not even continuous on $\mathbb{R}^\infty_c$ if $(c_n)_n \in \ell^2 \setminus \mathbb{R}^\infty_c$.

Part (ii)

As in (i), we show that $(S_N)_N$ converges in $L^2(X,\gamma)$. By a similar calculation as above and in the proof of Thm. 4.2.6 (p. 45), (remember, $\mathcal{X}_n(x) = \langle x, e_n \rangle$)

$$
\begin{aligned}
\|S_N - S_M\|_{L^2(X,\gamma)} &= \int_X \left| \sum_{k=M+1}^{N} c_k \mathcal{X}_k(x) \right|^2 \gamma(dx) \\
&= \sum_{k=M+1}^{N} c_k^2 \int_{\mathbb{R}} x_k^2 \, \mathcal{N}(0,\lambda_k)(dx_k) = \sum_{k=M+1}^{N} c_k^2 \lambda_k,
\end{aligned}
$$

where we used the decomposition $\mathcal{N}(0,Q) = \bigotimes_{k\in\mathbb{N}} \mathcal{N}(0,\lambda_k)$ with respect to the ONB $\{e_k : k \in \mathbb{N}\}$. Since by assumption $\sum_{n\in\mathbb{N}} c_n^2 \lambda_n < \infty$, we conclude that $(S_N)_N$ is an L^2-Cauchy sequence, hence L^2-convergent. By the arguments in the beginning, this implies that $(S_N)_N$ is even pointwise convergent γ-a.e. and consequently that f is a measurable linear functional.

[2] it is not hard to show that a sequence $(x^m)_m \subset \mathbb{R}^\infty$ converges in the metric d introduced in Chapter 4 if and only if $(x_n^{(m)})_m$ converges in $\mathbb{R}$ for every $n \in \mathbb{N}$.

Obviously, if $(c_n)_n \in \ell^2$, then $\sum_{n\in\mathbb{N}} c_n x_n \leq \|c_n\|_{\ell^2}\, \|x\|_H$ by Cauchy-Schwarz and the isometry $x \mapsto (x_n)_n$ from X to ℓ^2. Hence, in this case f is continuous from X to $\mathbb{R}$.
If $(c_n)_n \notin \ell^2$, choose

$$x_n^{(m)} = \left\{ \begin{array}{ll} c_n, & n \leq m \\ 0, & n > m \end{array} \right\}, \quad n, m \in \mathbb{N}.$$

Then, for every $m \in \mathbb{N}$, $(x_n^{(m)})_{n\in\mathbb{N}} \in \mathbb{R}_c^\infty$ and $\|(x_n^{(m)})_n\|_{\ell^2}^2 = \sum_{n=1}^m c_n^2$. On the other hand, $f((x_n^{(m)})_n) = \sum_{n=1}^m c_n^2$. Thus,

$$\frac{f(x^m)}{\|x^m\|_{\ell^2}} = \left(\sum_{n=1}^m c_n^2\right)^{\frac{1}{2}} \to \infty \text{ as } m \to \infty.$$

Hence, f is not continuous.

A theorem by P. Lévy

Since the notion of *independent* random variables was not mentioned in the lectures, we provide the reader with the definition.

Definition. A finite family $\mathcal{Y}$ of random variables from Ω to $\mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, P)$ is called *(mutually) independent* if for all $n \in \mathbb{N}$ and pairwise-distinct $\mathcal{Y}_1, .., \mathcal{Y}_n \in \mathcal{Y}$ it holds that

$$\forall y_1, .., y_n \in \mathbb{R} : \prod_{k=1}^n P(\{\mathcal{Y}_k \leq y_k\}) = P(\bigcap_{k=1}^n \{\mathcal{Y}_k \leq y_k\}).$$

Definition. A sequence $(\mathcal{Y}_n)_n$ of random variables $\mathcal{Y}_n : \Omega \to \mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, P)$ is called *independent*, if

$$\forall n \in \mathbb{N} : \quad \{\mathcal{Y}_k : 1 \leq k \leq n\} \text{ are independent random variables.}$$

Theorem 1. *(P. Lévy) Let $(\mathcal{Y}_n)_n$ be a sequence of independent random variables on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{S}_N = \sum_{k=1}^N \mathcal{Y}_k$. Then, the following assertions are equivalent.*

1. *$(\mathcal{S}_N)$ converges in probability (in measure).*
2. *$(\mathcal{S}_N)$ converges in pointwise almost-surely (pointwise P-a.e.).*

We refer to [2] for a proof (even for Banach space-valued random variables) and to [1] for the classical case[3].

References

[1] K.L. Chung. *A course in probability theory*, Third edition, Academic Press, Inc., San Diego, CA, 2001.

[2] K. Itô and M. Nisio. *On the convergence of sums of independent Banach space valued random variables*, Osaka J. Math. 5:35–48, 1968.

[3] We are thankful to Jürgen Voigt for pointing out the latter reference.